



# On a theorem of Rhemtulla on polycyclic groups

B.A.F. Wehrfritz

School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, England, United Kingdom

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## ABSTRACT

Let  $H$  be a subgroup of the polycyclic-by-finite group  $G$  and denote the automorphism group of  $G$  by  $\Gamma$ . We prove that there exists an integer  $d$  such that in the poset  $\{\bigcap_{\gamma \in \Sigma} H^\gamma : \Sigma \text{ a subset of } \Gamma\}$  of all intersections of images  $H^\gamma$  of  $H$  under  $\Gamma$ , chains have length at most  $d$ . In particular the poset satisfies the minimal condition. This extends and improves a theorem of A.H. Rhemtulla. We also provide a very different proof of Rhemtulla's theorem.

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Let  $H$  be any subgroup of the polycyclic-by-finite group  $G$ . Denote by  $\mathbf{L}(G, H)$  the poset of all intersections of conjugates of  $H$ ; that is

$$\mathbf{L}(G, H) = \left\{ \bigcap_{g \in S} H^g : S \text{ a subset of } G \right\},$$

ordered by inclusion. Rhemtulla in [3] proves, purely group theoretically, that  $\mathbf{L}(G, H)$  satisfies the minimal condition, a result used much later in an algorithmic calculation of the normal core  $H_G = \bigcap_{g \in G} H^g$  of  $H$  in  $G$ ; see for example [1], 9.2.26.

The following gives a completely different approach to Rhemtulla's theorem. By Proposition 1.3 of [8] we may regard  $G$  as a subgroup of  $GL(n, \mathbf{Z})$ , for some  $n \geq 1$  and  $\mathbf{Z}$ , the integers, such that  $H$  is (Zariski) closed in  $G$ . Then every element of  $\mathbf{L}(G, H)$  is closed in  $G$  and  $G$  satisfies the minimal condition on closed sets. Thus Rhemtulla's theorem follows. We can obtain slightly more. By Theorem 2.6 of [8] we can regard  $G$  as a subgroup of some  $GL(n, \mathbf{Z})$  in such a way that for some vector  $x$  in  $\mathbf{Z}^n$  we have  $H = C_G(x)$ . Then  $H^g = C_G(xg)$  and

$$\bigcap_{g \in S} H^g = C_G \left( \sum_{g \in S} \mathbf{Q}xg \right)$$

for  $\mathbf{Q}$  the rationals. It follows that every chain in  $\mathbf{L}(G, H)$  has length at most  $n$ .

Our objective in this paper is to use the above approach to study the automorphism group  $\Gamma = \text{Aut } G$  and the holomorph  $\Gamma G$  of  $G$ . Thus for  $H$  a subgroup of the group  $G$  set

$$\mathbf{L}(\Gamma, H) = \left\{ \bigcap_{\gamma \in \Sigma} H^\gamma : \Sigma \text{ a subset of } \Gamma \right\},$$

again ordered by inclusion. Clearly  $\mathbf{L}(G, H)$  is a subposet of  $\mathbf{L}(\Gamma, H)$ . The following is our main result.

**Theorem.** *Let  $H$  be a subgroup of the polycyclic-by-finite group  $G$  and set  $\Gamma = \text{Aut } G$ . Then there is a faithful  $\Gamma G$ -module  $V$  that is additively free abelian of finite rank, a  $G$ -submodule  $U$  of  $V$  with  $V/U$  additively free abelian and an element  $x$  of  $V$  satisfying*

$$H = C_G(x \bmod U) = \{g \in G : x(g - 1) \in U\}.$$

E-mail address: [b.a.f.wehrfritz@qmul.ac.uk](mailto:b.a.f.wehrfritz@qmul.ac.uk).

In this theorem  $H = G \cap (1 + C)$  where  $C = \{a \in \text{End}_{\mathbf{Q}}(\mathbf{Q}V) : xa \in \mathbf{Q}U\}$  is a  $\mathbf{Q}$ -subspace of  $E = \text{End}_{\mathbf{Q}}(\mathbf{Q}V) \cong \mathbf{Q}^{n \times n}$ , for  $n$  the rank of  $V$ . Let  $K \in \mathbf{L}(\Gamma, H)$ . Then there exists a subspace  $A$  of  $E$  such that  $K = G \cap (1 + A)$ . If  $B$  is a subspace of  $E$  with  $G \cap (1 + B) \supseteq K$ , then  $A \cap B$  is a subspace of  $E$  with  $K = G \cap (1 + (A \cap B))$ . Let  $A(K)$  denote the intersection of all the subspaces  $A$  as above. Clearly  $A(K)$  is a subspace of  $E$  with  $K = G \cap (1 + A(K))$ . Also if  $K \leq L \in \mathbf{L}(\Gamma, H)$ , then  $A(K) \leq A(L)$ . Thus chains in  $\mathbf{L}(\Gamma, H)$  have length at most  $n^2$ . Consequently we have the following two corollaries.

**Corollary 1.** *Let  $H$  be a subgroup of the polycyclic-by-finite group  $G$  and set  $\Gamma = \text{Aut } G$ . Then there is an integer  $d$  such that chains in the poset  $\mathbf{L}(\Gamma, H)$  have length at most  $d$ . In particular  $\mathbf{L}(\Gamma, H)$  satisfies the minimal condition and the characteristic core  $H_{\Gamma} = \bigcap_{\gamma \in \Gamma} H^{\gamma}$  of  $H$  in  $G$  is the intersection of at most  $d + 1$  of the  $H^{\gamma}$ .*

**Corollary 2.** *Let  $H$  be a subgroup of the group  $G$  with  $H^G = \langle H^g : g \in G \rangle$  polycyclic-by-finite. Then there is an integer  $d$  bounding the lengths of chains in  $\mathbf{L}(G, H)$  and  $H_G$  is the intersection of at most  $d + 1$  conjugates of  $H$  in  $G$ .*

Does there exist for our polycyclic-by-finite group  $G$  an integer  $e$  such that for every subgroup  $H$  of  $G$  the normal core  $H_G$  of  $H$  in  $G$  is the intersection of at most  $e$  conjugates of  $H$ ? If so is there a bound on the lengths of chains in  $\mathbf{L}(G, H)$  for fixed  $G$  but over all choices of  $H$ ? Probably not, but what is clear is that the existence of such bounds cannot be proved by our methods here, as the following shows.

There is an integer-valued function  $f(n)$  of  $n$  only such that every finite subgroup of  $GL(n, \mathbf{Q})$ , for  $\mathbf{Q}$  the rationals, has order dividing  $f(n)$ ; see [5], 9.33. Let  $G = \langle g \rangle$  be an infinite cyclic group and set  $H = \langle g^m \rangle$  for some  $m \geq 1$ . Suppose there exist  $G$ -modules  $U \leq V$  and an element  $x$  of  $V$  with  $V/U$  additively free abelian of rank at most  $n$  and  $H = C_G(x \bmod U)$ . Then  $g$  has order  $m$  on  $W = (\mathbf{Z}xG + U)/U$  and hence on  $\mathbf{Q}W$ . Thus  $m$  divides  $f(n)$ . Consequently  $(G : H) = m$  is bounded by a function of  $n$  only. More generally this argument proves the following.

Let  $H$  be a subgroup of the group  $G$  of finite index. Suppose there exist  $G$ -modules  $U \leq V$  and  $x \in V$  with  $V/U$  additively free abelian of rank at most  $n$  and  $H = C_G(x \bmod U)$ . Then  $G^{f(n)} \leq H$ . If also  $G$  is polycyclic-by-finite, then  $G/G^{f(n)}$  is finite and hence given  $n$  there are only finitely many possibilities for  $H$  with the above properties. Consequently for no infinite polycyclic-by-finite group  $G$  can we choose the  $V$ ,  $U$  and  $x$  in the theorem so as to bound the ranks of the  $V/U$  over all subgroups  $H$  of  $G$ , or even over just the subgroups  $H$  of  $G$  of finite index.

**Lemma.** *Let  $G, H$  and  $K$  be subgroups of a group  $L$  with  $H \leq G$  and the index  $(L : K)$  finite. Suppose there is a  $K$ -module  $V$  that additively is free abelian of finite rank  $r$ , a  $(G \cap K)$ -submodule  $U$  of  $V$  and an element  $x$  of  $V$  with  $H \cap K = C_{G \cap K}(x \bmod U)$ . Then there exists an  $L$ -module  $V_1$  that additively is free abelian of finite rank  $(L : K)r$ , a  $G$ -submodule  $U_1$  of  $V_1$  and an element  $x_1$  of  $V_1$  such that  $H = C_G(x_1 \bmod U_1)$ . If  $V/U$  is additively free abelian, then we may choose  $V_1/U_1$  to be additively free abelian and if  $U = \{0\}$ , we may choose  $U_1 = \{0\}$ .*

**Proof.** If  $x \in U$ , then  $H \cap K = G \cap K$ . In this case we may suppose that  $V$  is the trivial  $K$ -module  $\mathbf{Z}$  with  $U = \{0\}$  and  $x = 1$ . Thus in all cases we assume that  $x$  is chosen not to lie in  $U$ .

Let  $R$  be a right transversal of  $H \cap K$  to  $H$  containing 1. Then  $R$  lies in a right transversal  $S$  of  $G \cap K$  to  $G$  and  $S$  is contained in a right transversal  $T$  of  $K$  to  $L$ . Set

$$V_1 = V \otimes_{\mathbf{Z}K} \mathbf{Z}L = \bigoplus_{t \in T} V \otimes t \quad \text{and} \quad U_1 = \bigoplus_{s \in S} U \otimes s \leq V_1.$$

Let  $x_1 = \sum_{r \in R} x \otimes r \in V_1$ . If  $g \in G$  and  $s \in S$ , then  $sg = k_s s^{\sigma}$  for some unique  $k_s$  in  $G \cap K$  and permutation  $\sigma$  of  $S$ . Then  $U_1 g = \sum_{s \in S} U k_s \otimes s^{\sigma} \leq U_1$ . Thus  $U_1$  is a  $G$ -submodule of  $V_1$ . Clearly  $V_1$  has additive rank  $(L : K)r$  and

$$V_1/U_1 \cong_{\mathbf{Z}} (\bigoplus_{s \in S} (V/U) \otimes s) \oplus (\bigoplus_{t \in T \setminus S} V \otimes t),$$

so if  $V/U$  is additively free abelian, then so is  $V_1/U_1$ . Trivially  $U_1 = \{0\}$  if  $U = \{0\}$ .

If  $h \in H$  and  $r \in R$ , then  $rh = h_r r^{\rho}$  for some  $h_r$  in  $H \cap K$  and permutation  $\rho$  of  $R$ . Then

$$x_1 h = \sum_{r \in R} x h_r \otimes r^{\rho} \in \sum_{r \in R} x \otimes r^{\rho} + U_1 = x_1 + U_1.$$

Therefore  $H \leq C_G(x_1 \bmod U_1)$ .

Suppose  $g \in C_G(x_1 \bmod U_1)$ . Now  $g = kt$  uniquely for some  $k$  in  $G \cap K$  and  $t$  in  $S$ . As above for  $s \in S$  we have  $sg = k_s s^{\sigma}$ . Now  $1 \in S$ , so  $kt = g = k_1 1^{\sigma}$ ; hence  $k = k_1$  and  $t = 1^{\sigma}$ . Also  $S \supseteq R$ , so

$$\sum_{r \in R} x k_r \otimes r^{\sigma} = x_1 g \in x_1 + U_1 = (\bigoplus_{r \in R} (x + U) \otimes r) \oplus (\bigoplus_{s \in S \setminus R} U \otimes s).$$

If  $xk \in U$ , then  $x \in U k^{-1} = U$ , since  $k \in G \cap K$ , and yet we have arranged for  $x$  not to lie in  $U$ . Consequently  $xk \notin U$  and for some  $r \in R$  we have  $r = 1^{\sigma} = t$  and  $xk = xk_1 \in x + U$ . Hence  $t \in H$  and  $k \in C_{G \cap K}(x \bmod U) = H \cap K$ . Thus  $g = kt \in H$ . Therefore  $H = C_G(x_1 \bmod U_1)$ . The proof of the lemma is complete.  $\square$

**The proof of the Theorem.** The proof of 2.6 of [8] constructs a subgroup  $J$  of  $G$  (there called  $(L \cap G_2) (H \cap G_2)$ ); note that  $G_2$  can be chosen normal in  $G_1$  since  $N_1$  is normal in  $G_1$  of finite index,  $m$  say, in  $G$  such that  $J$  modulo its Fitting subgroup  $N$  is abelian, for which there exists a cyclic  $J$ -module  $Y = y\mathbf{Z}J$  that is additively free abelian of finite rank,  $r$  say, such that  $N$  acts unipotently on  $Y$  and  $H \cap J = C_J(y)$ . Replace  $J$  by  $G^{m!}$ . Thus now  $J$  is also a characteristic subgroup of  $G$ .

Let  $\mathbf{k}$  denote the kernel of the given (right) action of  $\mathbf{Z}J$  on  $Y$  and let  $\mathbf{n}$  be the ideal of  $\mathbf{Z}J$  generated by the subset  $N - 1$  of  $\mathbf{Z}J$ . Since  $N$  is normal in  $J$  and acts unipotently on  $Y$  we have  $Y\mathbf{n}^r = \{0\}$ . Thus  $(\mathbf{k} + \mathbf{n})^r \leq \mathbf{k}$ . Let  $T/(\mathbf{k} + \mathbf{n})^r$  denote the  $\mathbf{Z}$ -torsion submodule of  $\mathbf{Z}J/(\mathbf{k} + \mathbf{n})^r$  and set  $X = \mathbf{Z}J/T$  and  $x_X = 1 + T$ . Then  $TJ \leq T$  and  $X = x_X \mathbf{Z}J$  is a right  $J$ -module. Also  $Y$  is additively torsion-free, so  $\mathbf{Z}J/\mathbf{k}$  is additively torsion-free and  $T \leq \mathbf{k}$ . Therefore  $x_X \mapsto y$  determines a  $\mathbf{Z}J$ -homomorphism  $\phi$  of  $X$  onto  $Y$ . Note that additively  $X$  is free abelian of finite rank, since  $\mathbf{Z}J/(\mathbf{k} + \mathbf{n})^r$  is finitely  $\mathbf{Z}$ -generated by [5], Page 22, Point 1, or [9], 4.7.

Clearly  $\Gamma$  normalizes  $J$  and  $N$ . By Lemma 2.1 of [7] (or see [4], Page 95, Lemma 10) there is a normal subgroup  $\Delta$  of  $\Gamma$  of finite index with  $[J, \Delta] \leq N$ . Now  $\Delta$  acts on  $\mathbf{Z}J$  on the right via its action on  $J$  and this makes  $\mathbf{Z}J$  into a  $\Delta J$ -module, where  $\Delta J \leq \Gamma G$ , the holomorph of  $G$ . If  $g \in J$  and  $\delta \in \Delta$ , then

$$g^\delta = g[g, \delta] = g + g([g, \delta] - 1) \in g + \mathbf{n}.$$

Thus  $\Delta$  centralizes  $\mathbf{Z}J/\mathbf{n}$  and hence  $\Delta$  normalizes  $\mathbf{k} + \mathbf{n}$ ,  $(\mathbf{k} + \mathbf{n})^r$  and  $T$ . Consequently the action of  $\Delta J$  on  $\mathbf{Z}J$  makes  $X$  into a  $\Delta J$ -module.

Let  $U_X$  denote the kernel of  $\phi$ . Then  $U_X$  is a  $\mathbf{Z}J$ -submodule of  $X$  with  $X/U_X \cong Y$ . Since  $x_X \phi = y$  we have

$$H \cap J = C_J(y) = C_J(x_X \bmod U_X).$$

By our lemma above there is a  $\Gamma G$ -module  $V_1$  that is additively free abelian of finite rank, a  $G$ -submodule  $U$  of  $V_1$  with  $V_1/U$  additively free abelian and an element  $x$  of  $V_1$  such that  $H = C_G(x \bmod U)$ . Possibly  $\Gamma G$  does not act faithfully on  $V_1$ . However there does exist a faithful  $\Gamma G$ -module  $V_2$  that is additively free abelian of finite rank, by a theorem of Merzljakov (see [2] or [7], Corollary 1.4, or [4], Page 96, Theorem 7). Now set  $V = V_1 \oplus V_2$ . The proof of the theorem is complete.  $\square$

**Remark.** There is a sort of weak partial converse to our theorem. Let  $H$  be a soluble-by-finite subgroup of the subgroup  $G$  of  $GL(n, \mathbf{Z})$  and suppose that  $H$  has finite index in its (Zariski) closure  $K$  in  $G$ . For example, this is the case whenever  $H$  is unipotent-by-finite. Then there exists a faithful  $G$ -module  $V$  that additively is free abelian of finite rank and an element  $x$  of  $V$  such that  $H = C_G(x)$ .

To prove this, note that  $H$  is profinitely closed in  $GL(n, \mathbf{Z})$ , see [6], Theorem 2, or [4], Page 61, Theorem 5. Hence there is a subgroup  $J$  of  $G$  of finite index with  $J \cap K = H$ . Then  $H$  is closed in  $J$  and the remark follows from [8], 2.4, applied to  $H \leq J$  and our lemma above (or [8], 2.2, if you prefer) to lift the conclusion from  $H \leq J$  to  $H \leq G$ .

If  $G = GL(n, \mathbf{Z})$  in the remark, then  $(K : H)$  finite means that  $H$  is an arithmetic group (see [4], Page 119). Not every polycyclic group is isomorphic to an arithmetic group; see [4], Page 259, Proposition 3. Thus the restriction to  $(K : H)$  being finite in the remark restricts the possible isomorphism classes of polycyclic group that can arise here as the subgroup  $H$ .

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